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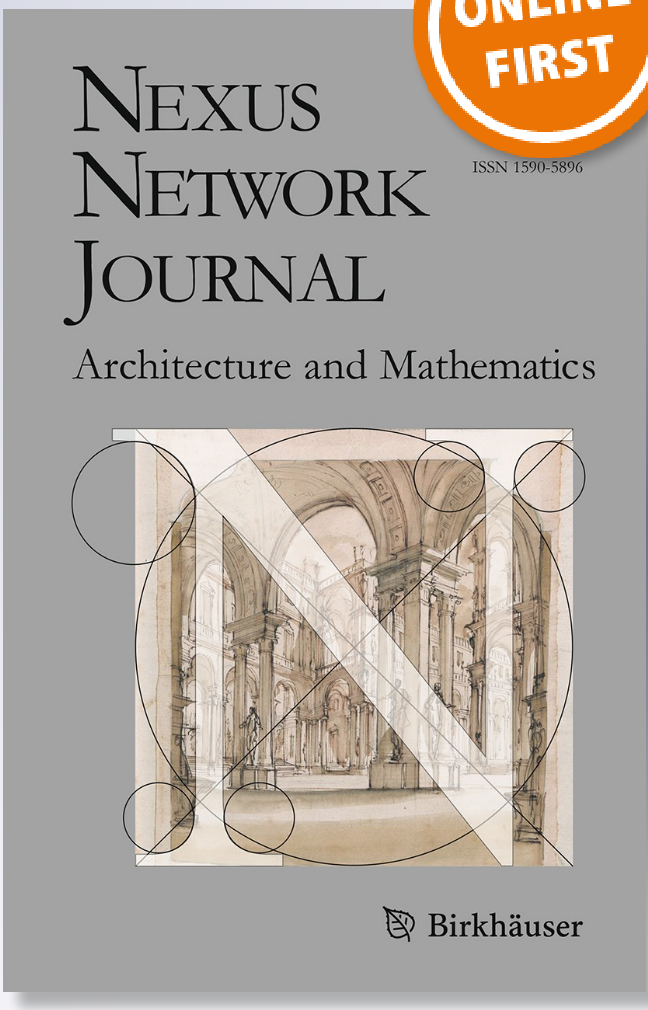

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RESEARCH

Ratios from the Intersections of 10 + 1 Proportionalities

Hyoung-June Park[1]

Abstract An innovative mathematical analysis comparing sets of preferred ratios from authors from antiquity (Vitruvius), the Renaissance (Alberti, Serlio and Palladio), and the modern age (Fechner and Lalo) with the eleven unique and universal proportionalities sheds new light on architects' use of certain ratios to endow their creations with commensurability and beauty. Some ratios may provide more ways of representing three magnitudes, and this might provide a clue to their enduring appearance in architectural works.

Keywords Proportion theory · Ratio · Proportionality · Commensurability

Introduction

Ratios have been used as guidelines for designers throughout the centuries as a means of transforming their geometries. While it is true that different people recommended and used different ratios, many of these ratios are similar.

A ratio is the relation of one number compared to the other, written as $p : q$ or p/q. Proportionality, also known as "mean", is the variety achieved by representing three ordered real numbers $(0 < x < y < z)$ and their differences $(z - y, y - x, z - x)$ with the equalities of the ratios among them. For example, $(z - y)/(y - x) = y/x$ or $(z - y) : (y - x) = y : x$. The equalities of the ratios are referred to as proportional balance or equilibrium within the three numbers and their differences. With these equalities, proportionality guarantees "commensurability" among them. "Mean" is commonly used to signify a specific method for finding a

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middle term when there are two extremes such as small and large term. In what follows, ways of representing the equalities of the ratios among the three ordered numbers and their differences are defined as "proportionality" (March 1998: 72–77) rather than "mean" in the modern definition.

This paper provides a mathematical account for preferred ratios and their relationships with various proportionalities.

Preferred Ratios

Based upon the Pythagorean and Platonic traditions, the treatments of ratio and their applications to design have been a central concern for intellectuals from the age of Humanism up until present day (March 1998, 1999; Mitrovic 1990; Fletcher 2001; Howard and Longair 1982; Heath 1921; Scholfield 1958; Huntley 1970).

It has been noticed by many scholars that several ratios were more frequently employed when designing rectangular room shapes, and were justified in the name of beauty (Scholfield 1958; Shin 1996; March 1998). To govern the shape of atriums, the Roman architect Vitruvius defines three classes of shape: the diagonal and side of a square, $\sqrt{2}:1$; a square and a half, 3:2; a square and two-thirds 5:3. Vitruvius also defines two classes of dining rooms: a double square, 2:1, and a square, 1:1:

> In width and length, atriums are designed according to three classes. The first is laid out by dividing the length into five parts and giving three parts to the width; the second, by dividing it into three and assigning two parts to the width; the third, by using the width to describe a square figure with equal sides, drawing a diagonal line in this square, and giving the atrium the length of this diagonal line. … Dining rooms ought to be twice as long as they are wide … But in the case of exedrae or square oeci… (Vitruvius 1960: 177, 179).

Leon Battista Alberti, who is credited with bringing musical theory and architectural proportion together, explains that his analogy of musical scales in architecture are developed from primary ratios like 1:1, 3:4, 2:3, 9:16,1:2, 4:9, 3:8, 1:3, 1:4.

> We have dealt with shorter areae, either with equal dimensions or with proportions of, say, two to three or three to four; and we have dealt with intermediate areae, where one dimension is twice the other or where the proportions are, say, four to nine or nine to sixteen. Finally we mentioned extended areae, with proportions of one to three, one to four, or, say, three to eight (Alberti 1994: 306).

These ratios are the successive lines of the Pythagorean tetraktys (March 1998).

Sebastiano Serlio provides another set of seven room ratios: the quadrate, 1:1; the sexquiquarta, 5:4; the sexquitertia, 4:3; the diagonea, $\sqrt{2}:1$; the sexquialtera, 3:2; the superbitienstertias, 5:3; and the dupla, 2:1.

> There are many quadrangle proportions, but I will here set down but seven of the principallest of them, which shall best serve for the use of a workman. First, this forme is called a right four cornered quadrate… (Serlio 1982: Bk I, Ch. I, fols. 11–12).

According to Andrea Palladio, the seven room shapes are: the circle; the square, 1:1; the diagonea, $\sqrt{2}:1$; the square and a third, 4:3; the square and a half, 3:2; the square and two-thirds, 5:3; and the double square, 2:1.

> There are seven types of room that are the most beautiful and well proportioned and turn out better: they can be made circular, though these are rare; or square; or their length will equal the diagonal of the square of the breadth; or a square and a third; or a square and a half; or a square and two-thirds; or two squares (Palladio 1997: 53).

Gustav Fechner in 1896 and Charles Lalo in 1908 (Lalo 1908) experimented the ratios in the area of psychophysics considering user preference (Huntley 1970: 64). They found ten different ratios used in man-made artifacts that were found to be preferred by their users. The ratios are 1:1, 5:6, 4:5, 3:4, 7:10, 2:3, 5:8, 13:23, 1:2, and 2:5 where 5:8 is the approximate natural number ratio of $1{:}(1+\sqrt{5})/2$ (Table 1).

In summary, Table 2 shows the preferred ratios of each of these authors.

Proportionalities

According to Aristotle (1924), the whole is greater than the sum of the parts (*Metaphysics* VIII.1045a), and the whole is some aggregate of the parts (*Metaphysics* V.1023b). The relationships between the parts and the whole are treated in mathematics as commensurability. Based upon the notion of ratio ($x{:}y$), the notion of proportion ($x{:}y{:}z$) is developed. The least set of numbers that can establish a proportion is 3. For three numbers $x, y, z,$ and $0 < x < y < z$, there are 3 possible outcomes of comparison, 1 unique case of equality, $x : y = y : z$, which is called geometric, and 2 cases of inequality, $x{:}y < y{:}z$ and $x{:}y > y{:}z$. For each case of inequality, there can be an infinite number of subcases with respect to the actual numbers involved in the comparison. Among these relationships, some are more significant than others when commensurability among three ordered terms and their differences is established when they have equality among their proportional relationships. For example, when there is the inequality $x{:}y < y{:}z$, commensurability among $x, y,$ and z, is not established. However, if $(1/z) - (1/y) = (1/y) - (1/x)$, the inequality $x : y < y : z$ can be rewritten as an equality, namely, $(z-y)/z = (y-x)/x$ (March 1998). Then, commensurability is established among x, y, and z in terms of $(z-y)$, z, $(y-x)$, and x. When the commensurability is achieved among them, we achieve equilibrium among the parts and the whole.

The inequality cases ($x{:}y < y{:}z$ and $x{:}y > y{:}z$) were treated by ancient Greek mathematicians over the centuries (Heath 1921; Nicomachus 1938; March 1998). With defining the arithmetic and the harmonic mean, Archytas showed the commensurability among x, y, and z is established in terms of $(z-y)$, $(y-x)$, and

Table 1 Fechner's and Lalo's measurements, after (Huntley 1970: 64)

Ratio width/length	Best rectangle		Worst rectangle	
	Fechner %	Lalo %	Fechner %	Lalo %
1.0 (1:1)	3.0	11.7	27.8	22.5
0.83 (5/6)	0.2	1.0	19.7	16.6
0.80 (4/5)	2.0	1.3	9.4	9.1
0.75 (3/4)	2.5	9.5	2.5	9.1
0.69 (7/10)	7.7	5.6	1.2	2.5
0.67 (2:3)	20.6	11.0	0.4	0.6
0.62 (5:8)	35.0	30.3	0.0	0.0
0.57 (13:23)	20.0	6.3	0.8	0.6
0.50 (1:2)	7.5	8.0	2.5	12.5
0.40 (2:5)	1.5	15.3	35.7	26.6
	100.0	100.0	100.0	100.1

Table 2 Aggregation of referred ratios

Vitruvius	Alberti	Serlio	Palladio	Fechner/Lalo
1:1	1:1	1:1	1:1	1:1
				$\cong$5:6
		4:5		4:5
	3:4	3:4	3:4	3:4
$1:\sqrt{2}$		$1:\sqrt{2}$	$1:\sqrt{2}$	$\cong$7:10
2:3	2:3	2:3	2:3	$\cong$2:3
3:5		3:5	3:5	$\cong$5:8
	9:16			$\cong$13:23
1:2	1:2	1:2	1:2	1:2
	4:9			
				2:5
	3:8			
	1:3			
	1:4			

$(z - y)$, $(y - x)$, z, and x. Eudoxus possibly added one subcontrary case to harmonic mean, and two subcontrary cases to geometric mean as shown in Table 4. Nicomachus and Pappus respectively added two distinct sets of 4 means with three overlapping cases among them. Thus, the total number of means for the inequalities becomes 10. These 10 means of inequality plus the first initial mean of equality, the geometric mean, brought the total number of means to 11 (10 + 1) (Heath 1921: 87). Among these 11 means, only 4 survive in current discourse: the initial 3, the geometric, arithmetic, and harmonic mean, and Fibonacci with its unique progression (Table 4).

Table 3 27 Possible combinations

(1) $\frac{z-y}{y-x}=\frac{x}{x}$	(8) $\frac{z-y}{y-x}=\frac{x}{z}$	(15) $\frac{z-x}{y-x}=\frac{z}{y}$	(22) $\frac{z-x}{z-y}=\frac{y}{y}$
(2) $\frac{z-y}{y-x}=\frac{y}{x}$	(9) $\frac{z-y}{y-x}=\frac{y}{z}$	(16) $\frac{z-x}{y-x}=\frac{z}{z}$	(23) $\frac{z-x}{z-y}=\frac{x}{y}$
(3) $\frac{z-y}{y-x}=\frac{z}{x}$	(10) $\frac{z-x}{y-x}=\frac{x}{x}$	(17) $\frac{z-x}{y-x}=\frac{x}{z}$	(24) $\frac{z-x}{z-y}=\frac{z}{y}$
(4) $\frac{z-y}{y-x}=\frac{y}{y}$	(11) $\frac{z-x}{y-x}=\frac{y}{x}$	(18) $\frac{z-x}{y-x}=\frac{y}{z}$	(25) $\frac{z-x}{z-y}=\frac{z}{z}$
(5) $\frac{z-y}{y-x}=\frac{x}{y}$	(12) $\frac{z-x}{y-x}=\frac{z}{x}$	(19) $\frac{z-x}{z-y}=\frac{x}{x}$	(26) $\frac{z-x}{z-y}=\frac{x}{z}$
(6) $\frac{z-y}{y-x}=\frac{z}{y}$	(13) $\frac{z-x}{y-x}=\frac{y}{y}$	(20) $\frac{z-x}{z-y}=\frac{y}{x}$	(27) $\frac{z-x}{z-y}=\frac{y}{z}$
(7) $\frac{z-y}{y-x}=\frac{z}{z}$	(14) $\frac{z-x}{y-x}=\frac{x}{y}$	(21) $\frac{z-x}{z-y}=\frac{z}{x}$	

It is worth noting that only 11 unique cases of proportionality exist among three ordered terms and their differences. For three real numbers x, y, z, and $0 < x < y < z$, the number of all possible combinations among x, y, z, and $(z-y)$, $(y-x)$, $(z-x)$, in terms of the equality of the ratios among them is 27. Table 3 shows the mathematical representations of the 27 possible cases in terms of the equalities of the ratios.

Among the 27 possible cases, 6 cases: (10), (13), (16), (19), (22), and (25), are easily eliminated because they violate the given condition, which is $0 < x < y < z$. Among the remaining 21 cases, the arithmetic mean arises in three equivalent ways: (1), (4), and (7), and the geometric mean arises in two equivalent ways: (2) and (6). This reduces the number of the cases to 16. Among the 16 cases, only 9 cases satisfy the given condition that we have real numbers x, y, z, and $0 < x < y < z$. Thus, the total number of the cases to satisfy the condition becomes

Table 4 11 proportionalities

Proportionality 1 (arithmetic) Definition: $\frac{z-y}{y-x}=1$	Proportionality 2 (geometric) Definition: $\frac{z-y}{y-x}=\frac{z}{y}$ or $\frac{z-y}{y-x}=\frac{y}{x}$
Proportionality 3 (harmonic) Definition: $\frac{z-y}{y-x}=\frac{z}{x}$	Proportionality 4 (subcontrary to harmonic) Definition: $\frac{z-y}{y-x}=\frac{x}{z}$
Proportionality 5 (subcontrary to geometric) Definition: $\frac{z-y}{y-x}=\frac{x}{y}$	Proportionality 6 (subcontrary to geometric) Definition: $\frac{z-y}{y-x}=\frac{y}{z}$
Proportionality 7 Definition: $\frac{z-x}{y-x}=\frac{z}{x}$	Proportionality 8 Definition: $\frac{z-x}{z-y}=\frac{z}{x}$
Proportionality 9 Definition: $\frac{z-x}{y-x}=\frac{y}{x}$	Proportionality 10 (Fibonacci) Definition: $\frac{z-x}{z-y}=\frac{y}{x}$
Proportionality 11 Definition: $\frac{z-x}{z-y}=\frac{z}{y}$	

When $0 < x < y < z$, and x, y, z are real numbers

11. This proves that the 11 proportionalities discovered by the ancient Greek mathematicians have remained both universal and unique ways for representing commensurability and proportional equilibrium among three ordered terms and their differences. Table 4 shows the 11 proportionalities and their properties (Heath 1921: 87).

Ratios from the Intersections of 11 Proportionalities

In order to observe the characters of the 11 proportionalities, their different patterns in growth are shown in Fig. 1. When the first term, x, is fixed at 1, the third term, z, was projected with the increase of the second term y from the first term based upon the calculation with the definition of each proportionality shown in Table 4.

From the delineation of their different growth patterns, the 15 intersections among the patterns are observed and identified as shown in Table 5.

Between $1 < y \leq (1 + \sqrt{2})$, y values at the 15 intersections among the 11 proportionalities were calculated with the definition of each proportionality in Table 4. $1{:}\sqrt{2}$ and $1{:}(1 + \sqrt{5})/2$ have multiple intersections at different z values so that 13 ratios between x and y from the observed 15 cases are finally identified as shown in Table 6. Between $y > (1 + \sqrt{2})$ and $y < 100$, no significant intersections among the 11 proportionalities are observed.

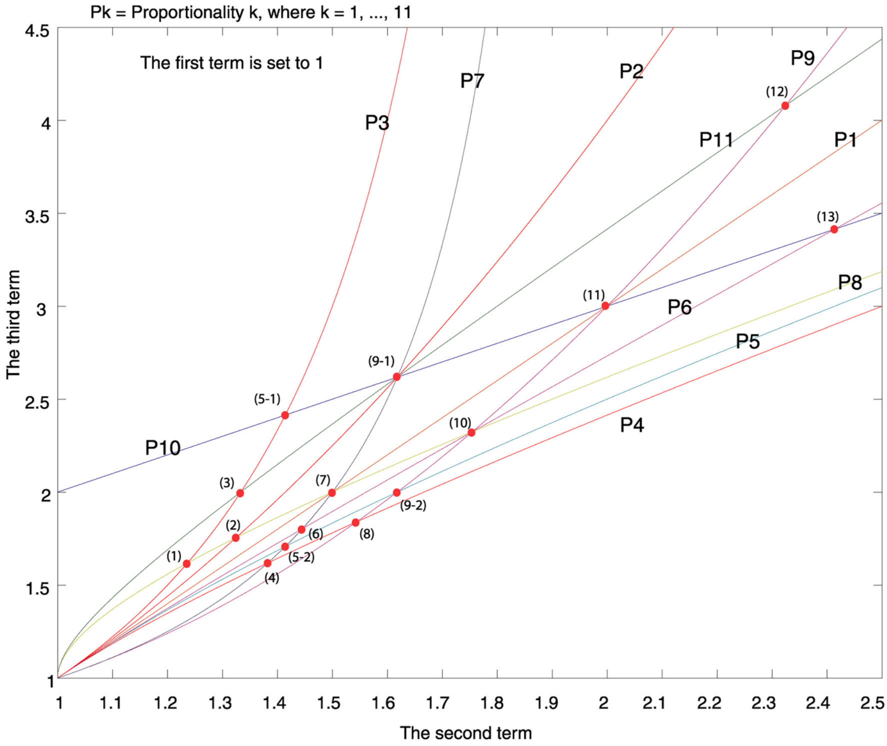

Fig. 1 Growth patterns of the 11 proportionalities

Table 5 15 intersections among the growth patterns of the 11 proportionalities

Intersection number	x	y	z	Crossed proportionalities
(1)	1	$-1+\sqrt{5}$	$\frac{1+\sqrt{5}}{2}$	Proportionality 3 and 8
(2)	1	$\cong 1.3247$	1.75488	Proportionality 2 and 8
(3)	1	$\frac{4}{3}$	2	Proportionality 3 and 11
(4)	1	$\frac{5-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	Proportionality 4 and 7
(5–1)	1	$\sqrt{2}$	$1+\sqrt{2}$	Proportionality 3 and 10
(5–2)	1	$\sqrt{2}$	$\frac{2+\sqrt{2}}{2}$	Proportionality 5 and 7
(6)	1	$\cong 1.44504$	1.80194	Proportionality 6 and 7
(7)	1	$\frac{3}{2}$	2	Proportionality 1, 7, and 8
(8)	1	$\cong 1.54369$	1.83929	Proportionality 4 and 9
(9–1)	1	$\frac{1+\sqrt{5}}{2}$	$\frac{3+\sqrt{5}}{2}$	Proportionality 2, 7, 10 and 11
(9–2)	1	$\frac{1+\sqrt{5}}{2}$	2	Proportionality 5 and 9
(10)	1	$\cong 1.75488$	2.32472	Proportionality 6, 8 and 9
(11)	1	2	3	Proportionality 1, 9 and 10
(12)	1	$\cong 2.32472$	4.0796	Proportionality 9 and 11
(13)	1	$1+\sqrt{2}$	$2+\sqrt{2}$	Proportionality 6 and 10

Table 6 13 ratios from the 15 intersections among the 11 proportionalities

x	y
1	$-1+\sqrt{5}$
1	$\cong 1.3247$
1	$\frac{4}{3}$
1	$\frac{5-\sqrt{5}}{2}$
1	$\sqrt{2}$
1	$\cong 1.44504$
1	$\frac{3}{2}$
1	$\cong 1.54369$
1	$\frac{1+\sqrt{5}}{2}$
1	$\cong 1.75488$
1	2
1	$\cong 2.32472$
1	$1+\sqrt{2}$

Comparisons

In Table 7 the preferred ratios of architects and designers given in Table 1 are compared to the 13 ratios identified from the 15 intersections of the growth patterns of the 11 proportionalities shown in Fig. 1.

Table 7 Comparisons between the 13 ratios from the 15 intersections of the 11 proportionalities and the preferred ratios; $x : y$ = the first term:the second term

$x : y$ from the 15 Intersections	Vitruvius	Alberti	Serlio	Palladio	Fechner and Lalo	Crossed proportionalities (prop.)	Number of crossed proportionalities
$1 : \left(-1+\sqrt{5}\right) \cong 4{:}5$			$\cong 4{:}5$		$\cong 4{:}5$	Prop. 3 and 8	2
$\cong 1{:}1.3247$						Prop. 2 and 8	2
$1 : \frac{4}{3} = 3 : 4$		3:4	3:4	3:4	3:4	Prop. 3 and 11	2
$1 : \frac{5-\sqrt{5}}{2}$						Prop. 4 and 7	2
$1 : \sqrt{2}$	$1 : \sqrt{2}$		$1 : \sqrt{2}$	$1 : \sqrt{2}$	$\cong 7{:}10$	When $z = 1 + \sqrt{2}$, Prop. 3 and 10 When $z = \frac{2+\sqrt{2}}{2}$, Prop. 5 and 7	(2) (2) 4
$\cong 1{:}1.44504$						Prop. 6 and 7	2
$1 : \frac{3}{2} = 2 : 3$	2:3	2:3	2:3	2:3	$\cong 2{:}3$	Proportionality 1, 7, and 8	3
$\cong 1{:}1.54369$						Proportionality 4 and 9	2
$1 : \frac{1+\sqrt{5}}{2} \cong 3{:}5$	$\cong 3{:}5$		$\cong 3{:}5$	$\cong 3{:}5$	$\cong 5{:}8$	When $z = \frac{3+\sqrt{5}}{2}$, Prop. 2, 7, 10 and 11 When $z = 2$, Prop. 5 and 9	(4) (2) 6
$\cong 1{:}1.75488$		$\cong 9{:}16$			$\cong 13{:}23$	Proportionality 6, 8 and 9	3
1:2	1:2	1:2	1:2	1:2	1:2	Proportionality 1, 9 and 10	3
$\cong 1{:}2.32472$		$\cong 4{:}9$				Proportionality 9 and 11	2
$1 : (1+\sqrt{2})$					$\cong 2{:}5$	Proportionality 6 and 10	2

It is worth noting that the 1:1 ratio between x and y does not satisfy the definition of proportionality such that $0 < x < y < z$. Out of the preferred ratios, one ratio (x:$y \cong$ 5:6) from Fechner/Lalo, and three ratios (x:y = 3:8, 1:3, 1:4) from Alberti's long rooms are not involved with any 13 ratios from the proportionality intersections. All the other preferred ratios are related to the ratios identified from the intersections of at least two different proportionalities. Out of the 13 ratios from the proportionality intersections, four ratios ($\cong$1:1.3247; 1:$(5 - \sqrt{5})/2$; $\cong$1:1.44504; and $\cong$1:1.54369) were not linked to any of the preferred ratios.

Discussion

Table 7 shows that certain ratios between x and y provide various ways of establishing commensurability among three ordered real numbers ($0 < x < y < z$) and their differences ($z - y$, $y - x$, $z - x$) when z is defined according to the 11 proportionalities. In other words, when x and y have a particular ratio, there is a higher probability that different proportionalities have the same z. This means that the particular ratios can provide more ways of representing three ordered numbers among the dimensions of an object and their differences with the equalities of the ratios among the ordered numbers. They provide the variety of commensurability for understanding or appreciating an object. This may hint at a possible underlying reason as to why certain ratios have been cherished and repeatedly employed in architecture and design throughout the ages.

References

Aristotle. 1924. *Metaphysics*. Trans. W. D. Ross. Oxford: The Clarendon Press.

Alberti, Leon Battista. 1994. *On the Art of Building in Ten Books*. Trans. J. Rykwert, N. Leach, R. Tavernor. Cambridge, MA: MIT Press.

Fletcher, Rachel. 2001. Palladio's Villa Emo: The Golden Proportion Hypothesis Defended. *Nexus Network Journal* **3**(2):105–112.

Heath, Thomas. 1921. *A History of Greek Mathematics*. Oxford: Clarendon Press.

Howard, Deborah and Malcolm Longair. 1982. Harmonic Proportion and Palladio's *Quattro Libri*. *Journal of the Society of Architectural Historians* **41**: 116–143.

Huntley, H.E. 1970. *The Divine Proportion: A Study in Mathematical Beauty*. New York: Dover Publications, Inc.

Lalo, Charles. 1908. *L'esthétique expérimentale contemporaine*. Paris: Alcan.

March, Lionel. 1998. *Architectonics of Humanism: Essays on Number in Architecture*. Los Angeles: Academy Editions.

March, Lionel. 1999. Architectonics of Proportion: Historical and Mathematical Grounds. *Environment and Planning B: Planning and Design* **26**: 447–454.

Mitrović, Branko. 1990. Palladio's Theory of Proportion and the Second Book of the *Quattro Libri dell'Architettura*. *Journal of the Society of Architectural Historians* **49**: 279–292.

Nicomachus of Gerasa. 1938. *Introduction to Arithmetic*. Trans. M. L. D'Ooge. Ann Arbor: University of Michigan.

Palladio, Andrea. 1997. *The Four Books on Architecture*. Trans. Robert Taverner and Richard Schofield. Cambridge, MA: MIT Press.

Scholfield, Paul H. 1958. *The Theory of Proportion in Architecture*. New York: Cambridge University Press.
Serlio, Sebastiano. 1982. *The Five Books of Architecture* (1611). New York: Dover.
Shin, Hyunho. 1996. Pythagorean Number Theory and Proportional Designs in the Second Book of Andrea Palladio's *I Quattro Libri dell'Architettura*. Doctoral Dissertation, Los Angeles, University of California,
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